Nichols Algebras and Nichols Systems

La Falda, July-August 2019

I. Heckenberger

Aims and Contents

In this course, crucial parts of the upcoming book *Hopf algebras and root systems*, joint with Hans-Jürgen Schneider, are presented. The main aim is to present a criterion (Corollary 8.5), based on reflections, for finite-dimensionality of the Nichols algebra of a semisimple Yetter-Drinfeld module, and to discuss the role of bosonization and right coideal subalgebras for this criterion. As a side result, a parametrization of the set of graded right coideal subalgebras of finite-dimensional Nichols algebras in terms of morphisms of the Weyl groupoid is given.

(1) Yetter-Drinfeld modules and the functor Ω changing module and comodule structures
(2) Braided Hopf algebras, gradings and Nichols systems
(3) Reflections of Nichols systems
(4) Right coideal subalgebras and their compatibility with reflections
(5) Tensor decompositions of Yetter-Drinfeld modules
(6) Excursion: Cartan graphs and their Weyl groupoids
(7) The semi-Cartan graph of a Nichols algebra
(8) The combinatorics of right coideal subalgebras
(9) Applications to Hopf algebras

Reflection is a tool for (braided) Hopf algebra triples. The theory is most efficient if several Hopf algebra triples are available.

1. Yetter-Drinfeld modules

Yetter-Drinfeld modules appear naturally in the context of Hopf algebra triples.

Definition 1.1. Let H be a Hopf algebra with bijective antipode. A **Yetter-Drinfeld module** V over H is a left H-module V with a left H-comodule structure

$\delta : V \to H \otimes V$ such that

$$\delta(hv) = h_{(1)}v_{(-1)}\mathcal{S}(h_{(3)}) \otimes h_{(2)}v_{(0)}$$

for all $h \in H$, $v \in V$, where $\delta(v) = v_{(-1)} \otimes v_{(0)}$.

Notation: ${}^H_H\mathcal{YD}$ = category of Yetter-Drinfeld modules over H.

Example 1.2. $H = \Bbbk G$, the group algebra of a group G. Then any H-comodule is of the form

$$V = \oplus_{g \in G} V_g$$

with $V_g = \{v \in V \mid \delta(v) = g \otimes v\}$ for all $g \in G$. Moreover, $V \in {}^H_H\mathcal{YD}$ for an H-module H-comodule V if and only if $hV_g = V_{hgh^{-1}}$ for all $g, h \in G$.

Until the end of the script let H be a Hopf algebra with bijective antipode $\mathcal{S}$.

Lemma 1.3. *If $V, W \in {}^H_H\mathcal{YD}$ then $V \oplus W \in {}^H_H\mathcal{YD}$ and $V \otimes W \in {}^H_H\mathcal{YD}$ with*

$$h(v \otimes w) = h_{(1)}v \otimes h_{(2)}w, \quad \delta(v \otimes w) = v_{(-1)}w_{(-1)} \otimes (v_{(0)} \otimes w_{(0)})$$

for all $h \in H$, $v \in V$, $w \in W$.

As a consequence, Yetter-Drinfeld modules over H form a monoidal category with morphisms being linear maps $f : V \to W$ which are H-module and H-comodule maps.

Proposition 1.4. *For any $V, W \in {}^H_H\mathcal{YD}$, the map $c_{V,W} : V \otimes W \to W \otimes V$,*

$$c_{V,W}(v \otimes w) = v_{(-1)}w \otimes v_{(0)}$$

*for all $v \in V$, $w \in W$, is an isomorphism in ${}^H_H\mathcal{YD}$, called the **braiding**. The inverse map is given by*

$$c^{-1}_{V,W}(w \otimes v) = v_{(0)} \otimes \mathcal{S}^{-1}(v_{(-1)})w.$$

Yetter-Drinfeld modules with their braiding form an important example of a braided monoidal category.

Finite-dimensional Yetter-Drinfeld modules admit natural duals. Indeed, if $V \in {}^H_H\mathcal{YD}$ is finite-dimensional, then the dual vector space V^* is in ${}^H_H\mathcal{YD}$ via

$$(hf)(v) = f(\mathcal{S}(h)v), \quad f_{(-1)}f_{(0)}(v) = \mathcal{S}^{-1}(v_{(-1)})f(v_{(0)})$$

for all $h \in H$, $f \in V^*$, $v \in V$.

There are several functors relating different categories of Yetter-Drinfeld modules. For us, the interchange of module and comodule structures will be crucial. The situation is most convenient if H is finite-dimensional.

Lemma 1.5. *Let $V \in {}^H_H\mathcal{YD}$. If H is finite-dimensional, then there is a unique Yetter-Drinfeld structure on V over $H^{*\mathrm{op}\,\mathrm{cop}}$ with*

$$fv = f(v_{(-1)})v_{(0)}, \quad hv = v_{[-1]}(h)v_{[0]}$$

for all $f \in H^*$, $h \in H$, $v \in V$, where $\delta(v) = v_{(-1)} \otimes v_{(0)}$ is the H-coaction and $\delta'(v) = v_{[-1]} \otimes v_{[0]}$ is the $H^{*\,\mathrm{cop}}$-coaction of V.

Remark 1.6. Typically, we will write $\Omega(V)$ for V with the Yetter-Drinfeld structure over $H^{*\mathrm{op}\,\mathrm{cop}}$.

With the definition, the H-comodule structure of V is turned into an $H^{*\,\mathrm{op}}$-module structure, and the H-module structure into an $H^{*\,\mathrm{cop}}$-comodule structure. The two Yetter-Drinfeld structures of V are equivalent in the sense that any one is obtained from the other. In mathematical terms: Ω is a braided monoidal isomorphism of categories. (Compatibilities with morphisms, tensor products, and with the braiding are available.)

If H is not finite-dimensional, then one has to deal with a dual pair (H, H') of Hopf algebras (where H' plays the role of $H^{*\mathrm{op}\,\mathrm{cop}}$) and one has to restrict himself to *rational* modules in order to preserve the equivalence.

The theory uses an even more general form of the equivalence Ω, where H is replaced by a braided Hopf algebra.

2. Nichols systems

The aim of this section is to introduce Nichols systems. This is our fundamental notion for the study of Nichols algebras of semi-simple Yetter-Drinfeld modules.

First we define braided Hopf algebras — here only Hopf algebras in the category of Yetter-Drinfeld modules.

Lemma 2.1. *Let $(D, \Delta_D, \varepsilon_D)$ and $(D', \Delta_{D'}, \varepsilon_{D'})$ be coalgebras in ${}^H_H\mathcal{YD}$, that is coalgebras, such that $D, D' \in {}^H_H\mathcal{YD}$ and $\Delta_D, \Delta_{D'}, \varepsilon_D, \varepsilon_{D'}$ are morphisms in ${}^H_H\mathcal{YD}$. Then $D \otimes D'$ is a coalgebra in ${}^H_H\mathcal{YD}$ with comultiplication*

$$\Delta = (\mathrm{id}_D \otimes c_{D,D'} \otimes \mathrm{id}_{D'})(\Delta_D \otimes \Delta_{D'}).$$

Definition 2.2. A tuple $(S, \mu, \eta, \Delta, \varepsilon, \mathcal{S})$ is a **Hopf algebra in ${}^H_H\mathcal{YD}$** with a Yetter-Drinfeld module S, multiplication μ, unit η, comultiplication Δ, counit ε, antipode $\mathcal{S}$, if all these maps are morphisms in ${}^H_H\mathcal{YD}$, and the usual Hopf algebra axioms are fulfilled.

Note that $\Delta_S = \Delta$ is an algebra map if and only if $\Delta_S \mu = (\mu \otimes \mu)\Delta_{S\otimes S}$. Thus the braiding of S is part of the definition of a Hopf algebra in ${}^H_H\mathcal{YD}$. If $H = \Bbbk$, then a Hopf algebra in ${}^H_H\mathcal{YD}$ is an ordinary Hopf algebra.

We will need $\mathbb{N}_0^\theta$-gradings, $\theta \in \mathbb{N}$, of Yetter-Drinfeld modules and of braided Hopf algebras. From the general perspective, often one can work with any abelian monoid rather than with $\mathbb{N}_0^\theta$.

Definition 2.3. Let Γ be an abelian monoid. A Yetter-Drinfeld module $V \in {}^H_H\mathcal{YD}$ is Γ-**graded**, if

$$V = \bigoplus_{\gamma \in \Gamma} V(\gamma)$$

for some subobjects $V(\gamma) \in {}^H_H\mathcal{YD}$. A morphism $f : V \to W$ between Γ-graded Yetter-Drinfeld modules V, W is **graded** if

$$f(V(\gamma)) \subseteq W(\gamma)$$

for all $\gamma \in \Gamma$.

If $V, W \in {}^H_H\mathcal{YD}$ are Γ-graded, then $V \otimes W$ is Γ-graded by

$$(V \otimes W)(\gamma) = \bigoplus_{\gamma' + \gamma'' = \gamma} V(\gamma') \otimes W(\gamma'')$$

for all $\gamma \in \Gamma$. Then the braiding $c_{V,W} : V \otimes W \to W \otimes V$ of Γ-graded Yetter-Drinfeld modules is a graded morphism.

Definition 2.4. Let Γ be an abelian monoid. A Hopf algebra $S \in {}^H_H\mathcal{YD}$ is Γ-**graded**, if $S \in {}^H_H\mathcal{YD}$ is Γ-graded and if all morphisms $\mu, \eta, \Delta, \varepsilon, \mathcal{S}$ are graded.

Let $\theta \in \mathbb{N}$ and $\mathbb{I} = \{1, 2, \dots, \theta\}$. We write $\mathcal{F}_\theta^H$ for the category of families

$$M = (M_1, \dots, M_\theta),$$

where $M_i \in {}^H_H\mathcal{YD}$ is finite-dimensional for each $i \in \mathbb{I}$. A morphism $f : M \to N$ in $\mathcal{F}_\theta^H$ is a tuple $f = (f_1, \dots, f_\theta)$, where $f_i : M_i \to N_i$ is a morphism in ${}^H_H\mathcal{YD}$ for all $i \in \mathbb{I}$.

In the most interesting cases, all coordinates M_i of our $M \in \mathcal{F}_\theta^H$ will be **irreducible** (i. e. non-zero and have no non-trivial Yetter-Drinfeld submodule), but for some constructions this assumption is not needed.

Let $(\alpha_i)_{i \in \mathbb{I}}$ denote the standard basis of $\mathbb{Z}^\theta$.

Definition 2.5. Let S be a Hopf algebra in ${}^H_H\mathcal{YD}$, $N_1, \dots, N_\theta$ be finite-dimensional subobjects of S in ${}^H_H\mathcal{YD}$, and $N = (N_1, \dots, N_\theta)$. Let

$$f = (f_j)_{j\in\mathbb{I}} : N \to M$$

be an isomorphism of tuples in $\mathcal{F}^H_\theta$ for some $M \in \mathcal{F}^H_\theta$. The triple $\mathcal{N} = \mathcal{N}(S, N, f)$ is called a **pre-Nichols system of** M if

(Sys1) S is generated as an algebra by $N_1, \dots, N_\theta$, and
(Sys2) S is an $\mathbb{N}_0^\theta$-graded Hopf algebra in ${}^H_H\mathcal{YD}$ with $\deg(N_j) = \alpha_j$ for all $j \in \mathbb{I}$.

Note that M seems to play a minor role in the above definition. The importance of M will turn out later, when out of one we construct infinitely many new pre-Nichols systems and relate them to each other.

Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of a tuple $M \in \mathcal{F}^H_\theta$. Then $S(0) = \Bbbk 1$ and $\sum_{j=1}^\theta N_j = \bigoplus_{j=1}^\theta N_j$ by (Sys1) and (Sys2). By a general result, the antipode of S is bijective. We will use the notation

$$\mathcal{N}_j = N_j, \quad 1 \le j \le \theta.$$

Definition 2.6. Let $M \in {}^H_H\mathcal{YD}$ be finite-dimensional (that is, $M \in \mathcal{F}^H_1$) and $\mathcal{N}(S, N, f)$ a pre-Nichols system of M. Then S is called a **pre-Nichols algebra of** M. A pre-Nichols algebra S of M is a **Nichols algebra of** M, if all primitive elements of S are in $S(1)$.

Remark 2.7. Up to isomorphism there is only one Nichols algebra of M. We write $\mathcal{B}(M)$ for it. For any pre-Nichols algebra S of M and any isomorphism $f : S(1) \to M$ there is a unique surjection $S \to \mathcal{B}(M)$ which is f on $S(1)$.

It is a general (usually very hard) problem to decide whether $\mathcal{B}(M)$ is finite-dimensional and to give a presentation of it by generators and relations.

Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of a tuple $M \in \mathcal{F}^H_\theta$. We write

$$p^{\mathcal{N}} : S \to \mathcal{B}(M)$$

for the surjective map of $\mathbb{N}_0^\theta$-graded Hopf algebras in ${}^H_H\mathcal{YD}$ which is defined by $f_j : N_j \xrightarrow{\cong} M_j \subseteq \mathcal{B}(M)$ on N_j, $j \in \mathbb{I}$. It is called the **canonical map of** $\mathcal{N}$.

For the definition of a Nichols system we need the braided adjoint action.

Definition 2.8. Let S be a Hopf algebra in ${}^H_H\mathcal{YD}$. The morphism

$$\mathrm{ad}_S = \mu(\mu \otimes \mathcal{S})(\mathrm{id}_S \otimes c_{S,S})(\Delta \otimes \mathrm{id}_S) : S \otimes S \to S$$

is called the **braided adjoint action**.

If S is a Hopf algebra, then $\mathrm{ad}_S(x \otimes y) = x_{(1)} y \mathcal{S}(x_{(2)})$ for all $x, y \in S$.

Definition 2.9. Let $M \in \mathcal{F}_\theta^H$, and $\mathcal{N} = \mathcal{N}(S, N, f)$ a pre-Nichols system of M. Let $i \in \mathbb{I}$. Then $\mathcal{N}$ is called a **Nichols system of** (M, i), if $p^{\mathcal{N}}$ defines bijective maps

(Sys3) $\Bbbk[N_i] \cong \mathcal{B}(M_i)$, and

(Sys4) $(\mathrm{ad}_S N_i)^n(N_j) \cong (\mathrm{ad}_{\mathcal{B}(M)} M_i)^n(M_j)$ for all $j \in \mathbb{I} \setminus \{i\}$ and $n \geq 0$.

Note that $\mathcal{N}_0 = \mathcal{N}(\mathcal{B}(M), M, \mathrm{id})$ is a Nichols system of (M, i) with canonical map $p^{\mathcal{N}_0} = \mathrm{id}_{\mathcal{B}(M)}$.

3. Reflections of Nichols systems

Definition 3.1. Let A be a Hopf algebra in ${}_H^H\mathcal{YD}$. A **Hopf algebra triple over** A is a triple (S, π, γ), where $\pi : S \to A$, $\gamma : A \to S$ are Hopf algebra morphisms in ${}_H^H\mathcal{YD}$ such that $\pi\gamma = \mathrm{id}_A$. The elements of the subalgebra

$$S^{\mathrm{co}A} = \{s \in S \mid (\mathrm{id} \otimes \pi)\Delta(s) = s \otimes 1\}$$

of S are called the **right coinvariants**.

Similarly, one defines the subalgebra ${}^{\mathrm{co}A}S$ of left coinvariants.

Hopf algebra triples are important because of the **bosonization**:

$$S \cong S^{\mathrm{co}A} \# A,$$

and $S^{\mathrm{co}A}$ is a braided Hopf algebra in ${}_{A\#H}^{A\#H}\mathcal{YD}$. Here, # means tensor product with a special multiplication and comultiplication, depending only on structures of $S^{\mathrm{co}A}$ and A. *(Notice that even if S and A are ordinary Hopf algebras, $S^{\mathrm{co}A}$ is a braided Hopf algebra. This is the main reason, why our generality is necessary.)*

The advantage of pre-Nichols systems is that they admit many natural Hopf algebra triples.

Definition 3.2. Let $i \in \mathbb{I}$, $M \in \mathcal{F}_\theta^H$, and let $\pi_i : \mathcal{B}(M) \to \mathcal{B}(M_i)$ be the Hopf algebra projection defined by the projection $\bigoplus_{j=1}^\theta M_j \to M_i$ in ${}_H^H\mathcal{YD}$.

Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. We write

$$\tilde{\pi}_i^{\mathcal{N}} : S \to \Bbbk[N_i], \quad \tilde{\gamma}_i^{\mathcal{N}} : \Bbbk[N_i] \to S$$

for the canonical $\mathbb{N}_0^\theta$-graded maps which are the identity on N_i. Moreover, let

$$K_i^{\mathcal{N}} = S^{\mathrm{co}\Bbbk[N_i]}, \quad L_i^{\mathcal{N}} = {}^{\mathrm{co}\Bbbk[N_i]}S,$$

where the left and right coinvariant elements are defined with respect to $\tilde{\pi}_i^{\mathcal{N}}$.

Remark 3.3. Let $i \in \mathbb{I}$ and let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. Then $(S, \tilde{\pi}_i^{\mathcal{N}}, \tilde{\gamma}_i^{\mathcal{N}})$ is a Hopf algebra triple over $\Bbbk[N_i]$.

In order to define reflections of Nichols systems, one needs an additional finiteness assumption.

Definition 3.4. Let $i \in \mathbb{I}$ and $N \in \mathcal{F}_\theta^H$. We say that N is i**-finite**, if for each $j \in \mathbb{I} \setminus \{i\}$ there exists $m \geq 0$ with $(\mathrm{ad}\, N_i)^m(N_j) = 0$ in $\mathcal{B}(N)$. In this case, let

$$a_{ii}^N = 2, \quad a_{ij}^N = -\max\{m \geq 0 \mid (\mathrm{ad}\, N_i)^m(N_j) \neq 0\}$$

for $j \neq i$, and $s_i^N \in \mathrm{Aut}(\mathbb{Z}^\theta)$ the reflection with

$$s_i^N(\alpha_j) = \alpha_j - a_{ij}^N \alpha_i$$

for all $j \neq i$. If N is i-finite, then the tuple $R_i(N) \in \mathcal{F}_\theta^H$ with

$$R_i(N)_i = N_i^*, \quad R_i(N)_j = (\mathrm{ad}\, N_i)^{-a_{ij}^N}(N_j)$$

for $j \neq i$ is called the i**-th reflection of** N.

Note that $s_i^N(\alpha_i) = -\alpha_i$ in the above definition.

Definition 3.5. Let $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a Nichols system of (M, i). Assume that M_j is irreducible for all $j \neq i$ and that M is i-finite. Let

$$\widetilde{N} = (\widetilde{N}_1, \ldots, \widetilde{N}_\theta), \quad \widetilde{f} = (\widetilde{f}_1, \ldots, \widetilde{f}_\theta), \quad R_i(\mathcal{N}) = \mathcal{N}(\widetilde{S}, \widetilde{N}, \widetilde{f}),$$

where $\widetilde{S} = \Omega(K_i^{\mathcal{N}}) \# \mathcal{B}(M_i^*)$ is the bosonization of the Hopf algebras $\Omega(K_i^{\mathcal{N}})$ and $\mathcal{B}(M_i^*)$, $\widetilde{N}_i = M_i^*$, $\widetilde{N}_j = (\mathrm{ad}\, N_i)^{-a_{ij}^N}(N_j)$ for $j \neq i$, $\widetilde{f}_i$ is the identity on M_i^*, and $\widetilde{f}_j : \widetilde{N}_j \to R_i(M)_j$ for $j \in \mathbb{I} \setminus \{i\}$ is the isomorphism induced by $p^{\mathcal{N}}$. The triple $R_i(\mathcal{N})$ is called the i**-th reflection of** $\mathcal{N}$.

The name *reflection* is justified partially by the following.

Proposition 3.6. *Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}_H^H\mathcal{YD}$ for all $j \in \mathbb{I} \setminus \{i\}$. Assume that M is i-finite. Let $\mathcal{N}$ be a Nichols system of (M, i). Then $R_i(\mathcal{N})$ is a Nichols system of $(R_i(M), i)$. Moreover,*

(1) *$R_i(M)_j$ is irreducible in ${}_H^H\mathcal{YD}$ for all $j \in \mathbb{I} \setminus \{i\}$ and $R_i(M)$ is i-finite,*
(2) *$a_{ij}^{R_i(M)} = a_{ij}^M$ for all $j \in \mathbb{I}$, and*
(3) *M and $R_i(R_i(M))$ are isomorphic $\mathcal{F}_\theta^H$.*

Interestingly, Nichols systems and their reflections are very closely related in many respects.

Proposition 3.7. *Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}_H^H\mathcal{YD}$ for all $j \in \mathbb{I} \setminus \{i\}$. Assume that M is i-finite. Then for any Nichols system $\mathcal{N}$ of (M,i) there is a unique algebra isomorphism*

$$T_i^{\mathcal{N}} : L_i^{R_i(\mathcal{N})} \to K_i^{\mathcal{N}}$$

in ${}_H^H\mathcal{YD}$ such that for all $j \in \mathbb{I} \setminus \{i\}$, $0 \le n \le -a_{ij}^M$, and $y \in (\mathrm{ad}_{\widetilde{S}} M_i^*)^n(\widetilde{N}_j)$,

$$T_i^{\mathcal{N}}(\mathcal{S}_{\widetilde{S}}^{-1}(y)) = -y.$$

This $T_i^{\mathcal{N}}$ restricts to an isomorphism

$$L_i^{R_i(\mathcal{N})}(\alpha) \to K_i^{\mathcal{N}}(s_i^{R_i(N)}(\alpha))$$

for all $\alpha \in \mathbb{N}_0^\theta$.

4. Right coideal subalgebras

Definition 4.1. A right coideal subalgebra C of a Hopf algebra $S \in {}_H^H\mathcal{YD}$ is a subalgebra in ${}_H^H\mathcal{YD}$ such that $\Delta(C) \subseteq C \otimes S$.

Generally, if C is a right coideal subalgebra of S in ${}_H^H\mathcal{YD}$, then $C \otimes H$ is a right coideal subalgebra of $S\#H$ containing H, but the converse is usually false.

Now we look at the compatibility of right coideal subalgebras with reflections.

Definition 4.2. For any $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and for any Nichols system $\mathcal{N} = \mathcal{N}(S,N,f)$ of (M,i) we define

$$\begin{aligned} \mathcal{K}(\mathcal{N}) &= \{E \mid E \subseteq S \ \mathbb{N}_0^\theta\text{-graded right coideal subalgebra in } {}_H^H\mathcal{YD}\},\\ \mathcal{K}_i^+(\mathcal{N}) &= \{E \mid E \in \mathcal{K}(\mathcal{N}),\, \mathcal{N}_i \subseteq E\},\\ \mathcal{K}_i^-(\mathcal{N}) &= \{E \mid E \in \mathcal{K}(\mathcal{N}),\, \mathcal{N}_i \nsubseteq E\}.\end{aligned}$$

Let $\mathcal{N} = \mathcal{N}(S,N,f)$ be a Nichols system of some (M,i) with $M \in \mathcal{F}_\theta^H$ and M_i irreducible. Then for an $\mathbb{N}_0^\theta$-graded right coideal subalgebra E of S, either $N_i \subseteq E$ or $E \subseteq L_i^{\mathcal{N}}$.

Theorem 4.3. *Let $M \in \mathcal{F}_\theta^H$, $i \in \mathbb{I}$, and let $\mathcal{N}$ be a Nichols system of (M,i). Assume that M is i-finite, and that M_j is irreducible in ${}_H^H\mathcal{YD}$ for all $j \in \mathbb{I}$.*

(1) *The map*

$$t_i^{\mathcal{N}}: \mathcal{K}_i^-(R_i(\mathcal{N})) \to \mathcal{K}_i^+(\mathcal{N}), \quad E \mapsto T_i^{\mathcal{N}}(E)\Bbbk[\mathcal{N}_i],$$

is bijective with inverse given by $E \mapsto (T_i^{\mathcal{N}})^{-1}(E \cap K_i^{\mathcal{N}})$.

(2) *The multiplication map* $T_i^{\mathcal{N}}(E) \otimes \Bbbk[\mathcal{N}_i] \to T_i^{\mathcal{N}}(E)\Bbbk[\mathcal{N}_i]$ *is bijective for all* $E \in \mathcal{K}_i^-(R_i(\mathcal{N}))$.

This theorem is the main ingredient towards a general construction of right coideal subalgebras. Indeed, let S be a finite-dimensional Nichols algebra in ${}^H_H\mathcal{YD}$ and let $E \neq \Bbbk 1$ be a right coideal subalgebra of S. Then E contains a non-zero primitive element, that is, a non-zero subspace of $S(1)$. Now the only thing to do is to find a context for S and E such that Theorem 4.3 applies. Then $E' = (t_i^N)^{-1}(E)$ is a right coideal subalgebra of another braided Hopf algebra, and $\dim(E') < \dim(E)$. By induction, E can be constructed using a finite sequence of reflections. Moreover, this procedure also gives a nice decomposition of E.

In fact, this idea applies not only to finite-dimensional Nichols algebras. However, the context has to be chosen carefully. Next we will specify the necessary notions: Tensor decompositions and Cartan graphs.

5. Tensor decompositions

Let us specify the idea from the previous section to decompose right coideal subalgebras.

Definition 5.1. Let V be an $\mathbb{N}_0^\theta$ graded object in ${}^H_H\mathcal{YD}$. We say that V is **tensor decomposable** if there exist $n \in \mathbb{N}_0$, irreducible objects $Q_1, \dots, Q_n \in {}^H_H\mathcal{YD}$ and pairwise distinct elements $\beta_1, \dots, \beta_n \in \mathbb{N}_0^\theta$ such that

$$V \cong \mathcal{B}(Q_1) \otimes \mathcal{B}(Q_2) \otimes \cdots \otimes \mathcal{B}(Q_n)$$

as $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$ with $\deg(Q_i) = \beta_i$ for all $1 \leq i \leq n$.

There exist more general notions of tensor decomposability, but for this course (and also for the book) the above definition is sufficient.

We will be interested in tensor decomposable Nichols algebras and right coideal subalgebras. Our aim is to provide more details on the data implementing a tensor decomposition. In important cases, for example if S is a finite-dimensional Nichols algebra in ${}^H_H\mathcal{YD}$ of a semi-simple Yetter-Drinfeld module, then S and all its right coideal subalgebras will be tensor decomposable.

In general, the existence of a tensor decomposition of an $\mathbb{N}_0^\theta$-graded object in ${}^H_H\mathcal{YD}$ is not guaranteed. However, a direct consequence of the theorem of Krull-Remak-Schmidt implies uniqueness of the tensor decomposition up to isomorphism. In particular, the number n and the set $\{\beta_1,\dots,\beta_n\}$ are uniquely determined.

6. Cartan graphs

We start with a purely combinatorial structure and then we discuss when a pre-Nichols system has a canonical Cartan graph.

For a finite set I, let $(\alpha_i)_{i\in I}$ be the standard basis of $\mathbb{Z}^I=\{f:I\to\mathbb{Z}\}$. Recall that a Cartan matrix is an integer valued matrix C with $c_{ii}=2$ for all i, $c_{ij}\le 0$ for all $i\ne j$, and such that $c_{ij}=0$ if and only if $c_{ji}=0$.

Definition 6.1. Let I be a non-empty finite set (the **labels**), $\mathcal{X}$ a non-empty set (the **points**), and $r:I\times\mathcal{X}\to\mathcal{X}$, $A:I\times I\times\mathcal{X}\to\mathbb{Z}$ maps. We write

$$r_i(X)=r(i,X)\in\mathcal{X},\quad a_{ij}^X=A(i,j,X)\in\mathbb{Z},\quad A^X=(a_{kl}^X)_{k,l\in I}\in\mathbb{Z}^{I\times I}$$

for all $i,j\in I$, $X\in\mathcal{X}$. We say that $\mathcal{G}=\mathcal{G}(I,\mathcal{X},r,A)$ is a **semi-Cartan graph** if for all $X\in\mathcal{X}$, A^X is a Cartan matrix, and

(CG1) for all $i\in I$, $r_i^2(X)=X$, and

(CG2) for all $i,j\in I$ and $X\in\mathcal{X}$, $a_{ij}^X=a_{ij}^{r_i(X)}$. (A^X and $A^{r_i(X)}$ have the same i-th row.)

The cardinality of I is the **rank of** $\mathcal{G}$. For all $i\in I$ and $X\in\mathcal{X}$ let $s_i^X\in\mathrm{Aut}(\mathbb{Z}^I)$ with

$$s_i^X(\alpha_j)=\alpha_j-a_{ij}^X\alpha_i$$

for all $j\in I$. This is a reflection of $\mathbb{Z}^I$.

One possible interpretation of a semi-Cartan graph is that for each point X one has a Cartan matrix A^X, and for each label $i\in I$, an i-neighbor $r_i(X)$ of X, such that (CG1) and (CG2) hold.

It is easy to write down semi-Cartan graphs with few points. In general, the Cartan matrices in different points do not coincide.

If there is only one point, then a semi-Cartan graph is the same as a Cartan matrix. In Lie theory, there is a Weyl group and a set of real roots attached to a Cartan matrix. Semi-Cartan graphs have a Weyl groupoid and real roots, and if the structure of real roots is good, we talk about a Cartan graph.

Definition 6.2. For a set $\mathcal{X}$ and a finite set I, let $\mathcal{D}(\mathcal{X},I)$ be the category with objects the elements of $\mathcal{X}$ such that

$$\mathrm{Hom}(X,Y)=\{(X,f,Y)\in\mathcal{X}\times\mathrm{End}(\mathbb{Z}^I)\times\mathcal{X}\}$$

and $(X,f,Y)\circ(Y,g,Z)=(X,fg,Z)$ for all $X,Y,Z\in\mathcal{X}$, $f,g\in\mathrm{End}(\mathbb{Z}^I)$.

For a semi-Cartan graph $\mathcal{G}=\mathcal{G}(I,\mathcal{X},r,a)$ let $\mathcal{W}(\mathcal{G})$ be the smallest subcategory of $\mathcal{D}(\mathcal{X},I)$ containing all objects from $\mathcal{X}$ and all morphisms $(r_i(X),s_i^X,X)$ with $X\in\mathcal{X}$, $i\in I$.

By abuse of notation we also write s_i^X for the morphism $(r_i(X),s_i^X,X)$ in $\mathcal{W}(\mathcal{G})$. By (CG1) and (CG2), $s_i^{r_i(X)}s_i^X=\mathrm{id}_{\mathbb{Z}^i}$ for all $X\in\mathcal{X}$ and $i\in I$. Thus the category $\mathcal{W}(\mathcal{G})$ is a groupoid (i. e. all morphisms are isomorphisms).

Definition 6.3. Let $\mathcal{G}=\mathcal{G}(I,\mathcal{X},r,A)$ be a semi-Cartan graph. For all $X\in\mathcal{X}$, the set

$$\boldsymbol{\Delta}^{X\,\mathrm{re}}=\{w(\alpha_i)\in\mathbb{Z}^I\mid w\in\mathrm{Hom}_{\mathcal{W}(\mathcal{G})}(Y,X),Y\in\mathcal{X},i\in I\}$$

is called the **set of real roots of $\mathcal{G}$ at** X. The real roots α_i, $i\in I$, are called **simple**. Real roots in $\mathbb{N}_0^I$ are called **positive**, those in $-\mathbb{N}_0^I$ **negative**.

The semi-Cartan graph $\mathcal{G}$ is **finite**, if $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ is finite for all $X\in\mathcal{X}$.

For all $X\in\mathcal{X}$ and $i,j\in I$ let

$$m_{ij}^X=|\boldsymbol{\Delta}^{X\,\mathrm{re}}\cap(\mathbb{N}_0\alpha_i+\mathbb{N}_0\alpha_j)|.$$

The semi-Cartan graph $\mathcal{G}$ is a **Cartan graph**, if

(CG3) for all $X\in\mathcal{X}$, $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ consists only of positive and negative roots, and

(CG4) for all $X\in\mathcal{X}$, $i,j\in I$ with $m_{ij}^X<\infty$, $(r_ir_j)^{m_{ij}^X}(X)=X$.

So far, there exists a classification of all finite Cartan graphs, but only by a computer calculation. Finite Cartan graphs are closely related to simplicial arrangements. Rank two Cartan graphs can be described naturally via triangulations of regular n-gons.

Morphisms in $\mathcal{W}(\mathcal{G})$ for a Cartan graph $\mathcal{G}$ are products of simple reflections s_i^X. They admit a theory very similar to the theory of Coxeter groups. In particular, for any point of a finite Cartan graph there is a unique longest morphism ending in that point.

We introduce two other axioms, which arise in the study of right coideal subalgebras.

Definition 6.4. Let $\mathcal{G}(I,\mathcal{X},r,A)$ be a semi-Cartan graph, $X\in\mathcal{X}$, $l\geq 0$, and $\kappa=(i_1,\dots,i_l)\in I^l$.

(1) For all $1\le k\le l$ let
$$\beta_k^{X,\kappa}=\mathrm{id}_X s_{i_1}\cdots s_{i_{k-1}}(\alpha_{i_k}).$$
Let $\Lambda^X(\kappa)=\{\beta_k^{X,\kappa}\mid 1\le k\le l\}$.

(2) We say that κ is X-**reduced**, if for all $1\le k\le l$, $\alpha_{i_k}\notin\Lambda^{r_{i_k}\cdots r_{i_1}(X)}(i_{k+1},\dots,i_l)$.

A sequence $\kappa\in I^l$ is X-reduced if and only if $\beta_p^{X,\kappa}\neq-\beta_q^{X,\kappa}$ for any $1\le p<q\le l$. Thus, any sequence in I^1 is X-reduced, and a sequence in I^2 is X-reduced if and only if it is not constant. For any semi-Cartan graph $\mathcal{G}(I,\mathcal{X},r,A)$ and any $X\in\mathcal{X}$, $i,j\in I$, let $\overline{m}_{ij}^X$ be the largest positive integer m such that the sequence $(i,j,i,j,\dots)$ of length m is X-reduced. If such an m doesn't exist, let $\overline{m}_{ij}^X=\infty$.

Proposition 6.5. *A semi-Cartan graph $\mathcal{G}(I,\mathcal{X},r,A)$ is Cartan if and only if it satisfies the following axioms.*

(CG3') *For any $X\in\mathcal{X}$ and any X-reduced sequence κ, $\Lambda^X(\kappa)\subseteq\mathbb{N}_0^I$.*

(CG4') *For any $X\in\mathcal{X}$ and $i,j\in I$ with $i\neq j$ and $\overline{m}_{ij}^X<\infty$ we have*
$$(r_ir_j)^{\overline{m}_{ij}^X}(X)=X,\quad \mathrm{id}_X(s_is_j)^{\overline{m}_{ij}^X}(\alpha_k)=\alpha_k$$
for all $k\in I\setminus\{i,j\}$.

In this case,
$$\boldsymbol{\Delta}^{X\,\mathrm{re}}\cap\mathbb{N}_0^I=\bigcup_{\kappa\ X\text{-reduced}}\Lambda^X(\kappa)$$
and $\overline{m}_{ij}^X=m_{ij}^X$ for all $X\in\mathcal{X}$ and $i,j\in I$.

7. The semi-Cartan graph of a Nichols algebra

Definition 7.1. Let $M\in\mathcal{F}_\theta^H$ such that M_j is irreducible for all $j\in\mathbb{I}$. Let $l\in\mathbb{N}_0$ and $i_1,\dots,i_l\in\mathbb{I}$. Let $\mathcal{N}$ be a pre-Nichols system of M.

(1) We say that M **admits the reflection sequence** $(i_1,\dots,i_l)$ if $l=0$ or if M is i_1-finite and $R_{i_1}(M)$ admits the reflection sequence $(i_2,\dots,i_l)$.

(2) We say that $\mathcal{N}$ **admits the reflection sequence** $(i_1,\dots,i_l)$ if $l=0$ or if $\mathcal{N}$ is a Nichols system of (M,i_1), M is i_1-finite, and $R_{i_1}(\mathcal{N})$ admits the reflection sequence $(i_2,\dots,i_l)$.

(3) We say that M **admits all reflections** if M admits all reflection sequences $(j_1, \dots, j_k)$ with $k \in \mathbb{N}_0$ and $j_1, \dots, j_k \in \mathbb{I}$.
(4) We say that $\mathcal{N}$ **admits all reflections** if $\mathcal{N}$ admits all reflection sequences $(j_1, \dots, j_k)$ with $k \in \mathbb{N}_0$ and $j_1, \dots, j_k \in \mathbb{I}$.
(5) Assume that M admits all reflections. Let

$$\mathcal{F}_\theta^H(M) = \{R_{j_1}(\cdots R_{j_k}(M)) \,|\, k \in \mathbb{N}_0, j_1, \dots, j_k \in \mathbb{I}\}.$$

Theorem 7.2. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible for all $j \in \mathbb{I}$. Assume that M admits all reflections. Let $\mathcal{X} = \{[P] \mid P \in \mathcal{F}_\theta^H(M)\}$, and let $r : \mathbb{I} \times \mathcal{X} \to \mathcal{X}$, $(i, [P]) \mapsto [R_i(P)]$. Then*

$$\mathcal{G}(M) = \mathcal{G}(\mathbb{I}, \mathcal{X}, r, (A^X)_{X \in \mathcal{X}}),$$

where $A^{[P]} = (a_{ij}^P)_{i,j \in \mathbb{I}}$ for all $[P] \in \mathcal{X}$, is a semi-Cartan graph.

Definition 7.3. Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible for all $j \in \mathbb{I}$. Assume that M admits all reflections. We call $\mathcal{G}(M)$ the **semi-Cartan graph of** M, and $\mathcal{W}(M) = \mathcal{W}(\mathcal{G}(M))$ the **Weyl groupoid of** M. Often it will be more convenient to say that $\mathcal{G}(M)$ is the semi-Cartan graph of $\mathcal{B}(M)$ and $\mathcal{W}(M)$ is the Weyl groupoid of $\mathcal{B}(M)$.

There is a special case where M admits all reflections.

Proposition 7.4. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible for all $j \in \mathbb{I}$. Assume that $\mathcal{B}(M)$ is a finite-dimensional vector space over $\Bbbk$. Then M admits all reflections, and $\dim \mathcal{B}(P) = \dim \mathcal{B}(M)$ for each $P \in \mathcal{F}_\theta^H(M)$.*

8. The combinatorics of right coideal subalgebras

In Theorem 4.3 we described a relationship between graded right coideal subalgebras of a Nichols system and its reflection, respectively. Now we are able to formulate an iterated version of this relationship.

Definition 8.1. Let $M \in \mathcal{F}_\theta^H$ such that M_i is irreducible in ${}^H_H\mathcal{YD}$ for all $i \in \mathbb{I}$. Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. Let $l \in \mathbb{N}_0$ and let $i_1, \dots, i_l \in \mathbb{I}$. Assume that $\mathcal{N}$ admits the reflection sequence $(i_1, \dots, i_l)$. Let

$$R_{()}(\mathcal{N}) = \mathcal{N}, \quad L_{()}^{\mathcal{N}} = S, \quad T_{()}^{\mathcal{N}} = \mathrm{id}_S, \quad \mathcal{K}_{()}^-(\mathcal{N}) = \mathcal{K}(\mathcal{N}), \quad t_{()}^{\mathcal{N}} = \mathrm{id}_{\mathcal{K}(\mathcal{N})},$$

and for any $1 \le k \le l$ define inductively

$$
\begin{aligned}
R_{(i_1,\dots,i_k)}(\mathcal{N}) &= R_{i_k}(\cdots R_{i_1}(\mathcal{N})),\\
L^{\mathcal{N}}_{(i_1,\dots,i_k)} &= \Big(T_{i_k}^{R_{(i_1,\dots,i_{k-1})}(\mathcal{N})}\Big)^{-1}\Big(K_{i_k}^{R_{(i_1,\dots,i_{k-1})}(\mathcal{N})} \cap L^{\mathcal{N}}_{(i_1,\dots,i_{k-1})}\Big),\\
T^{\mathcal{N}}_{(i_1,\dots,i_k)} &= T^{\mathcal{N}}_{i_1} T_{i_2}^{R_{i_1}(\mathcal{N})} \cdots T_{i_k}^{R_{(i_1,\dots,i_{k-1})}(\mathcal{N})} : L^{\mathcal{N}}_{(i_1,\dots,i_k)} \to S
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{K}^-_{(i_1,\dots,i_k)}(R_{(i_1,\dots,i_k)}(\mathcal{N})) =\\
&\Big(t_{i_k}^{R_{(i_1,\dots,i_{k-1})}(\mathcal{N})}\Big)^{-1}\Big(\mathcal{K}^+_{i_k}\big(R_{(i_1,\dots,i_{k-1})}(\mathcal{N})\big) \cap \mathcal{K}^-_{(i_1,\dots,i_{k-1})}\big(R_{(i_1,\dots,i_{k-1})}(\mathcal{N})\big)\Big),\\
&t^{\mathcal{N}}_{(i_1,\dots,i_k)} = t^{\mathcal{N}}_{i_1} \cdots t_{i_k}^{R_{(i_1,\dots,i_{k-1})}(\mathcal{N})} : \mathcal{K}^-_{(i_1,\dots,i_k)}(R_{(i_1,\dots,i_k)}(\mathcal{N})) \to \mathcal{K}(\mathcal{N}).
\end{aligned}
$$

Theorem 8.2. *Let $M \in \mathcal{F}^H_\theta$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections. Let $\mathcal{N} = \mathcal{N}(S, N, f)$ be a pre-Nichols system of M. Let $l \ge 1$ and $i_1, \dots, i_l \in \mathbb{I}$. Assume that $(i_1, \dots, i_l)$ is $[M]$-reduced in the semi-Cartan graph $\mathcal{G}(M)$ and that $\mathcal{N}$ admits the reflection sequence $(i_1, \dots, i_l)$. For any $1 \le k \le l$, let $\beta_k = \mathrm{id}_{[M]} s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$.*

1. *$\beta_1, \dots, \beta_l$ are pairwise distinct non-zero elements of $\mathbb{N}_0^\theta$.*
2. *For any $1 \le k \le l$, $R_{(i_1,\dots,i_{k-1})}(\mathcal{N})_{i_k} \subseteq L^{\mathcal{N}}_{(i_1,\dots,i_{k-1})}$. Let*
$$N_{\beta_k} = N^{\mathcal{N}}_k(i_1, \dots, i_l) = T^{\mathcal{N}}_{(i_1,\dots,i_{k-1})}(R_{(i_1,\dots,i_{k-1})}(\mathcal{N})_{i_k}).$$
3. *$\Bbbk 1 \in \mathcal{K}^-_{(i_1,\dots,i_l)}(R_{(i_1,\dots,i_l)}(\mathcal{N}))$. Let $E^{\mathcal{N}}(i_1, \dots, i_l) = t^{\mathcal{N}}_{(i_1,\dots,i_l)}(\Bbbk 1)$.*
4. *For any $1 \le k \le l$, $N_{\beta_k} \subseteq E^{\mathcal{N}}(i_1, \dots, i_l)$ is a finite-dimensional irreducible subobject in ${}^H_H\mathcal{YD}$ of degree β_k.*
5. *For any $1 \le k \le l$, the identity on N_{β_k} induces a graded isomorphism $\mathcal{B}(N_{\beta_k}) \cong \Bbbk[N_{\beta_k}] \subseteq S$ of $\mathbb{N}_0^\theta$-graded algebras in ${}^H_H\mathcal{YD}$.*
6. *The multiplication map $\Bbbk[N_{\beta_l}] \otimes \cdots \otimes \Bbbk[N_{\beta_1}] \to E^{\mathcal{N}}(i_1, \dots, i_l)$ is an isomorphism of $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$.*
7. *Let $E^{\mathcal{B}(M)}(i_1, \dots, i_l) = E^{\mathcal{N}_0}(i_1, \dots, i_l)$ with $\mathcal{N}_0 = \mathcal{N}(\mathcal{B}(M), M, \mathrm{id})$. The canonical map $p^{\mathcal{N}} : S \to \mathcal{B}(M)$ in ${}^H_H\mathcal{YD}$ induces an isomorphism*
$$E^{\mathcal{N}}(i_1, \dots, i_l) \to E^{\mathcal{B}(M)}(i_1, \dots, i_l).$$

In particular, all right coideal subalgebras $E^{\mathcal{N}}(i_1, \dots, i_l)$ in the Theorem are tensor decomposable, and precise information on the tensor decomposition is available.

Corollary 8.3. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$ and M admits all reflections. Then $\mathcal{G}(M)$ is a Cartan graph.*

Note that (CG3)' follows directly from the above theorem. The proof of (CG4)' needs a bit more effort.

Corollary 8.4. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Let $l \geq 1$ and $i_1, \dots, i_l \in \mathbb{I}$. Assume that M admits all reflections and that $\kappa = (i_1, \dots, i_l)$ is $[M]$-reduced in the semi-Cartan graph $\mathcal{G}(M)$. If $\alpha_i \in \Lambda^{[M]}(\kappa)$ for all $i \in \mathbb{I}$, then the following hold.*

(1) *$E^{\mathcal{B}(M)}(i_1, \dots, i_l) = \mathcal{B}(M)$.*
(2) *For any pre-Nichols system $\mathcal{N} = \mathcal{N}(S, N, f)$ of M admitting the reflection sequence κ, the map $p^{\mathcal{N}} : S \to \mathcal{B}(M)$ is bijective.*

Corollary 8.5. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in I$. The following are equivalent.*

(1) *$\mathcal{B}(M)$ is finite-dimensional*
(2) *M admits all reflections and*
    (a) *$\mathcal{G}(M)$ is finite, and*
    (b) *$\mathcal{B}(P_i)$ is finite-dimensional for all $P \in \mathcal{F}_\theta^H(M)$ and $i \in \mathbb{I}$.*

The latter corollary is the starting point of the classification of finite-dimensional Nichols algebras of semisimple Yetter-Drinfeld modules over finite groups.

Finally, we can give a description of right coideal subalgebras of finite-dimensional Nichols algebras of semi-simple Yetter-Drinfeld modules.

Corollary 8.6. *Let $M \in \mathcal{F}_\theta^H$ such that M_j is irreducible in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. Assume that M admits all reflections, and that $\mathcal{G}(M)$ is finite. For all $P \in \mathcal{F}_\theta^H(M)$, the map*

$$E^{\mathcal{B}(P)} : \mathrm{Hom}(\mathcal{W}(M), [P]) \to \mathcal{K}(\mathcal{B}(P)), \quad w \mapsto E^{\mathcal{B}(P)}(w),$$

is bijective, where $E^{\mathcal{B}(P)}(w) = E^{\mathcal{B}(P)}(\kappa)$ for any reduced decomposition κ of w,

In fact, there is a natural ordering (the Duflo or weak order) on $\mathrm{Hom}(\mathcal{W}(M), [P])$, such that the bijection in the Corollary is compatible with the inclusion on $\mathcal{K}(\mathcal{B}(P))$ and with this ordering on $\mathrm{Hom}(\mathcal{W}(M), [P])$.

9. Applications to Hopf algebras

The following problems have accessible proofs based on the presented methods.

(1) Theorem of Angiono: All finite-dimensional pointed complex Hopf algebras with abelian coradical are generated by skew-primitive elements.
(2) Classification of finite-dimensional coradically graded pointed complex Hopf algebras with abelian coradical
(3) Classification of finite-dimensional Nichols algebras over finite groups (under some technical restrictions to exclude the Nichols algebras of ireducible Yetter-Drinfeld modules)
(4) Quantum groups of finite type for non-roots of unity are Drinfeld doubles of Nichols algebras.
(5) Small quantum groups of finite type are Drinfeld doubles of Nichols algebras under some extra assumption on q. (We say: the braiding matrix is **genuinely of Cartan type**.) The latter is needed because in some cases the Nichols algebras have extra relations. The root vector relations of the small quantum groups can be obtained by the introduction of root vector sequences based on sequences of right coideal subalgebras. Such root vector sequences are then unique up to scalar multiples, and do not depend on ad hoc constructions.